PG – 216

Second Semester M.Sc. Examination, June 2016
(CBCS)
MATHEMATICS
M 201T : Algebra – II

Time : 3 Hours   Max. Marks : 70

Instructions : 1) *Answer **any five** questions.*
2) ***All** questions carry **equal** marks.*

1. a) Show that a Nil radical N(A) is an ideal of a ring A. Further, prove that $N\left(A/N(A)\right)=0$.

   b) Define the radical of an ideal r(I) of a ring A. For any ideals I and J of a ring A, prove that

   i) $I \subseteq J \Rightarrow r(I) \subseteq r(J)$ ii) $r(r(I)) = r(I)$

   c) Define extension and contraction of ideals with respect to a ring.

   If C denote the set of all contracted ideals I of A and E denote the set of all extended ideals J of B, then show that $C = \{ I : I^{ec} = I\}$ and $E = \{J : J^{ce} = J\}$. (4+5+5)

2. a) Show that every abelian group G is a module over the ring of integers.

   b) State and prove the first isomorphism theorem for modules.

   c) Define a finitely generated free A-module. Prove the following :

   i) Submodule of finitely generated modules need not be finitely generated.

   ii) The quotients of finitely generated module is finitely generated. (4+5+5)

3. a) Show that an A-module M is simple if and only if $M \cong A/I$ for some maximal ideal I of A.

   b) State and prove Schur's lemma for simple A-module.

   c) Define the Noetherian and Artinian modules. Write an example for the following :

   i) Both Noetherian and Artinian   ii) Noetherian but not Artinian. (4+6+4)

4. a) Define the modules of finite length. Show that a module is of finite length if and only if it is both Noetherian and Artinian.

   b) Define the Noetherian ring. If A is Noetherian ring, prove that the polynomial ring A[x] is Noetherian. (7+7)

P.T.O.

5. a) Define an algebraic element in an extension of a field.

   Let K be an extension of a field F. Prove that an element $a \in K$ is algebraic over F if and only if F(a) is a finite extension of F.

   b) Let R be the field of real numbers and Q, the field of rationals. Prove the following :

   i) $a = \sqrt{2}$, $b = \sqrt{3}$ are algebraic of degree 2 over Q

   ii) $Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3})$

   iii) $Q(\sqrt{2} + \sqrt{3})$ is algebraic extension of Q of degree 4 **(8+6)**

6. a) Show that a polynomial of degree 'n' over a field F can have atmost 'n' roots in any extension field.

   Is the result true, when extension field K is not a field F ? Justify with an example.

   b) Define the splitting field of a polynomial over a field F. Determine the splitting field of $x^2 + x + 1$ over the field Q. **(8+6)**

7. a) Show that the set $\mathcal{C}$ of all constructible number is a countable field.

   b) Prove that a regular pentagon is constructible by using edge and compass.

   c) Let $f(x) \in F(x)$ be irreducible. If f(x) has multiple roots, then show that the characteristic of F is not zero. **(5+5+4)**

8. a) Define a normal extension of a field. If K is a finite extension of a field F, then show that G(K, F) is a finite group and its order satisfies $O(G(K,F)) \leq [K:F]$.

   b) Show that K is a normal extension of a field F of characteristic 0 if and only if K is a splitting field of same polynomial over F. **(7+7)**